

# Three-body resonances $\Lambda nn$ and $\Lambda\Lambda n$

V. B. Belyaev<sup>1,2</sup>

<sup>1</sup>*Laboratory of Theoretical Physics, JINR, Dubna, 141980, Russia*

S. A. Rakityansky<sup>1,2</sup>

<sup>2</sup>*Department of Physics, University of South Africa, P.O. Box 392, Pretoria 0003, South Africa*

W. Sandhas<sup>3</sup>

<sup>3</sup>*Physikalisches Institut, Universitat Bonn, D-53115 Bonn, Germany*

(Dated: February 2, 2008)

Possible bound and resonant states of the hypernuclear systems  $\Lambda nn$  and  $\Lambda\Lambda n$  are sought as zeros of the corresponding three-body Jost functions calculated within the framework of the hyperspherical approach with local two-body S-wave potentials describing the  $nn$ ,  $\Lambda n$ , and  $\Lambda\Lambda$  interactions. Very wide near-threshold resonances are found for both three-body systems. The positions of these resonances turned out to be sensitive to the choice of the  $\Lambda n$ -potential. Bound  $\Lambda nn$  and  $\Lambda\Lambda n$  states only appear if the two-body potentials are multiplied by a factor of  $\sim 1.5$ .

PACS : 13.75.Ev, 21.80.+a, 21.45.+v, 25.70.Ef

Keywords : Lambda-nucleon potential, hypernuclei, three-body resonance, hyperspherical harmonics, Jost function

## I. INTRODUCTION

The  $\Lambda$ -hyperon belongs to a wide class of particles that are not in abundance in this world and therefore are not freely available for scattering experiments. The properties of their interaction with other particles are studied indirectly. For example, the most important and established way of studying the  $\Lambda N$  interaction consists in measuring and calculating the spectral properties of the so called  $\Lambda$ -hypernuclei (see, for example, Refs. [1, 2] and references therein), which are bound states of  $\Lambda$ -particles inside atomic nuclei. The most convenient for this purpose are very light nuclei with  $A \lesssim 10$ . Firstly, because such simple systems have simple spectra with only few well separated levels, and secondly, because they allow a reliable theoretical modelling based on rigorous few-body methods.

The hyperon-nucleon attraction is insufficient to bind a  $\Lambda N$  pair. The simplest hypernucleus is therefore the hypertriton  ${}^3_\Lambda\text{H}$ , i.e. a bound  $\Lambda pn$  complex. Its binding energy is very small (the  $\Lambda$  particle is separated at  $\sim 0.15$  MeV) [3, 4, 5]. So, it looks like a deuteron core surrounded by a  $\Lambda$ -halo [3, 5].

Similarly to traditional (non-strange) nuclear physics, where the deuteron is the first testing ground for any  $NN$  potential, the system  $\Lambda NN$  is used to constrain new models of the hyperon-nucleon interaction. This system was recently analyzed in Refs. [5, 6] using rigorous three-body equations with the potentials constructed within the constituent quark model. The authors of Ref. [5] gave another proof that the coupling between the  $\Lambda NN$  and  $\Sigma NN$  channels is very important for the hypertriton binding and showed that  ${}^3_\Lambda\text{H}$  is the only bound state of the  $\Lambda NN$  system. Their comprehensive analysis lacks only one thing: they did not consider possible three-body resonances. Meanwhile their results give a strong indi-

cation that such resonances may exist and be located not far from the threshold energy. Indeed, they found that the channel  $\Lambda nn$  is attractive but not sufficient to produce a bound state, and the curve for its Fredholm determinant turns towards zero near the threshold energy (see Fig. 4 of Ref. [5]). In our present paper, we partly fill in the gap by considering the  $\Lambda nn$  resonance state.

The  $\Lambda N$  and  $\Lambda\Lambda$  potentials are usually constructed in such a way that the calculations with these potentials reproduce experimentally known bound states of the hypernuclei. Unfortunately, it is very difficult to do scattering experiments with the  $\Lambda$ -particles because of their short lifetime ( $\sim 10^{-10}$  sec) and extremely low intensity of the beams that can be obtained.

It is well known that even when scattering data are available in full, it is impossible to construct an interaction potential in a unique way. One can always obtain different but phase-equivalent potentials (see, for example, Ref. [7]). In this respect the  $\Lambda N$ -case is beyond any hope since only few experimental points for the  $\Lambda p$  scattering are available [8, 9]. During the decades of studying the hypernuclei many features of the  $\Lambda N$ -interaction have been revealed. However the comparison of the theoretical and experimental spectra remains inconclusive. Different potentials lead to almost the same spectra of the hypernuclei. We therefore need an additional tool for testing the potentials.

In principle, such a tool could be based on studying the  $\Lambda$ -nucleus resonances, if they do exist [10, 11]. Indeed, while the scattering and bound states mostly reflect the on-shell properties of the interaction, the resonances strongly depend on its off-shell characteristics, which may be different for phase-equivalent potentials.

Our present work is an attempt to attract the attention of both theoreticians and experimentalists to the

low-energy resonances in the  $\Lambda$ -nuclear systems. As an example, we consider the three-body systems  $\Lambda nn$  and  $\Lambda\Lambda n$  in the minimal approximation,  $[L] = [L_{\min}]$ , of the hyperspherical harmonics approach. By locating the  $S$ -matrix poles on the second (unphysical) sheet of the complex energy surface, we show that these systems have near-threshold resonant states. The position of the poles turns out to be strongly dependent on the choice of the  $\Lambda N$ -potential. This fact supports the idea that the studying of the  $\Lambda$ -nucleus resonances could be very important for finding an adequate  $\Lambda N$ -potential.

The demands for adequate hyperon-nucleon ( $YN$ ) and hyperon-hyperon ( $YY$ ) potentials come not only from nuclear physics itself, but also from astrophysics. The studies of the neutron stars show that these very dense and compact objects are in fact “giant hypernuclei” (see, for example, Ref. [12] and references therein). The  $\Lambda$ -particles appear inside neutron stars when the density becomes approximately two times higher than the ordinary nuclear density. The equation of state, describing a neutron star, involves all the inter-particle potentials and therefore its solutions depend on their properties. In particular, the strength of the short-range repulsion in the pairs  $\Lambda N$  and  $\Lambda\Lambda$  is crucial for determining the maximum mass and size of a neutron star. The repulsive nature of the  $\Lambda nn$  three-body force (if it is indeed repulsive) would lead to additional stability of neutron stars. Moreover, the two-body  $YY$  interactions regulate the cooling behaviour of massive neutron stars [12].

So, the studies of hypernuclear systems are not only important for reaching a better understanding of the physics of strange particles, but may also have an important impact on some other branches of science. This is why the research in this field is carried on by many theoretical groups and experimental laboratories.

## II. THREE-BODY JOST FUNCTION

There are several different ways of locating quantum resonances. The most adequate are the methods based on the rigorous definition of resonances as the  $S$ -matrix poles at complex energies. This definition is universal and applicable to the systems involving more than just two colliding particles. Of course, the problem of locating the  $S$ -matrix poles is not an easy task, and especially for few-body systems. There are different approaches to this problem. To the best of our knowledge, so far only one of them has been applied to study the hyperon-nucleus resonant states. This was done in Ref. [10] using an analytic continuation of the rigorous three-body equations proposed by Alt, Grassberger, and Sandhas [13] and known as the AGS-equations. In our present paper, we follow a different approach based on direct calculation of the Jost function using the method suggested in Ref. [14].

The three-body systems we consider in the present paper, namely,  $\Lambda nn$  and  $\Lambda\Lambda n$ , do not have bound states in any of the two-body subsystems  $nn$ ,  $\Lambda n$ , or  $\Lambda\Lambda$ . The

only possible collision process for them is therefore the  $3 \rightarrow 3$  scattering. The wave functions describing the systems that cannot form clusters behave asymptotically as linear combinations of the incoming and outgoing hyperspherical waves (see, for example, Ref. [15]). Thus it is convenient to describe the three-body configuration using the hyperspherical coordinates, among which only one (the hyperradius) runs from zero to infinity while all the others (the hyperangles) vary within finite ranges.

Within the hyperspherical approach, the wave function is expanded in an infinite series over the hyperspherical harmonics (similarly to the partial wave decomposition in the two-body problem), and we end up with an infinite system of coupled hyperradial equations, which is truncated in practical calculations. All the details of the hyperspherical approach can be found, for example, in the review by M. Fabre de la Ripelle [16].

It should be noted that although the two-body potentials and masses for the three-body systems  $\Lambda nn$  and  $\Lambda\Lambda n$  are different, they can be treated using exactly the same equations. Indeed, in both of these systems, we have two identical particles with spin  $1/2$  and a third particle of the same spin. In what follows, we therefore consider a general system of this type.

Let  $m_1$  be the mass of one of the identical particles, and  $m_2$  be the mass of the third particle. Then the total mass of the system is  $M = 2m_1 + m_2$  and the reduced masses for the identical pair and for the third particle are  $\mu_1 = m_1/2$  and  $\mu_2 = 2m_1m_2/M$ , respectively. With the Jacobi coordinates shown in Fig. 1, the three-body Schrödinger equation can be written as

$$\left( \partial_r^2 + \frac{5}{r} \partial_r - \frac{1}{r^2} \mathcal{L}^2 + k^2 - V \right) \Psi_{\vec{k}_1, \vec{k}_2}^{[s]}(\vec{r}_1, \vec{r}_2) = 0, \quad (1)$$

where

$$V = 2M(U_{12} + U_{13} + U_{23}) \quad (2)$$

is the sum of the two-body potentials  $U_{ij}$ , the vectors  $\{\vec{k}_1, \vec{k}_2\}$  represent the incident momenta of the three-body collision along the corresponding configuration vectors  $\{\vec{r}_1, \vec{r}_2\}$ , the superscript  $[s] = ((s_1 s_2) s_{12} s_3) s \sigma$  denotes the spin quantum numbers for the spin-addition scheme  $\vec{s} = (\vec{s}_1 + \vec{s}_2) + \vec{s}_3$ , the variable

$$r = \sqrt{r_1^2 + r_2^2}, \quad (3)$$

is the hyperradius that gives the “collective” size of the system,  $k$  is related to the total energy,  $k^2 = 2ME$ , and can be called the hypermomentum, and the operator  $\mathcal{L}^2$  absorbs all the angular variables. It is defined as

$$\mathcal{L}^2 = -\frac{\partial^2}{\partial \alpha^2} - 4 \cot(2\alpha) \frac{\partial}{\partial \alpha} + \frac{1}{\cos^2 \alpha} \vec{\ell}_{\vec{r}_1}^2 + \frac{1}{\sin^2 \alpha} \vec{\ell}_{\vec{r}_2}^2 \quad (4)$$

with  $\alpha = \arctan(r_2/r_1)$ ,  $0 \leq \alpha \leq \pi/2$ , and  $\vec{\ell}_{\vec{r}_i}$  being the operators of the angular momenta associated with

the corresponding Jacobi coordinates. The solutions of the eigenvalue problem

$$\mathcal{L}^2 Y_{[L]}(\omega) = L(L+4)Y_{[L]}(\omega) \quad (5)$$

are the so called hyperspherical harmonics that depend on the hyperangles  $\omega = \{\Omega_{\vec{r}_1}, \Omega_{\vec{r}_2}, \alpha\}$  including the spherical angles  $\Omega_{\vec{r}_i}$  of the vectors  $\vec{r}_i$  and the angle  $\alpha$  that determines the ratio  $r_2/r_1$ . The subscript  $[L]$  is the multi-index  $[L] = \{L, \ell_1, \ell_2, \ell, m\}$  that includes the grand orbital quantum number,

$$L = \ell_1 + \ell_2 + 2n, \quad n = 0, 1, 2, \dots, \quad (6)$$

as well as the angular momenta associated with the Ja-

cobi vectors and the total angular momentum  $\ell$  together with its third component  $m$ . Combining  $Y_{[L]}(\omega)$  with the spin states  $\chi_{[s]} = |(s_1 s_2) s_{12} s_3) s \sigma\rangle$ , we obtain the functions

$$\Phi_{[L]}^{jj_z}(\omega) = \sum_{m\sigma} \langle \ell m s \sigma | j j_z \rangle Y_{[L]}(\omega) \chi_{[s]} \quad (7)$$

that constitute a full ortho-normal set of states with a given total angular momentum  $j$  in the spin-angular subspace.

Similarly to the two-body partial wave decomposition, we can expand a solution of Eq. (1) in the infinite series over the hyperspherical harmonics,

$$\Psi_{\vec{k}_1, \vec{k}_2}^{[s]}(\vec{r}_1, \vec{r}_2) = \frac{1}{r^{5/2}} \sum_{[L][L']jj_z} u_{[L][L']}^{jj_z}(E, r) \Phi_{[L]}^{jj_z}(\omega_{\vec{r}}) \Phi_{[L']}^{jj_z*}(\omega_{\vec{k}}), \quad (8)$$

where the hyperangle sets  $\omega_{\vec{r}}$  and  $\omega_{\vec{k}}$  are associated with the pairs  $\{\vec{r}_1, \vec{r}_2\}$  and  $\{\vec{k}_1, \vec{k}_2\}$ , respectively. After substituting this expansion into Eq. (1) and doing the projection onto the functions  $\Phi_{[L]}^{jj_z}$ , we end up with the following system of hyperradial equations

$$\left[ \partial_r^2 + k^2 - \frac{\lambda(\lambda+1)}{r^2} \right] u_{[L][L']} = \sum_{[L'']} V_{[L][L'']} u_{[L'']}[L'] , \quad (9)$$

where for the sake of simplicity we dropped the superscripts  $jj_z$  (indicating the conserving total angular momentum). In Eq. (9),

$$V_{[L][L']}(r) = 2M \int \Phi_{[L]}^{jj_z*}(\omega) (U_{12} + U_{13} + U_{23}) \Phi_{[L']}^{jj_z}(\omega) d\omega, \quad (10)$$

and  $\lambda = L + 3/2$ . Since we consider a system that cannot form clusters, the asymptotic behaviour of its wave function may only involve the incoming and outgoing hyperspherical waves  $\sim \exp(\mp ikr)$ , which are the products of the corresponding spherical waves along the Jacobi radii  $r_1$  and  $r_2$ ,

$$e^{ik_1 r_1} e^{ik_2 r_2} = e^{ikr \cos^2 \alpha} e^{ikr \sin^2 \alpha} = e^{ikr}.$$

We therefore look for the solution of matrix equation (9) as

$$u_{[L][L']}(E, r) = h_{\lambda}^{(-)}(kr) F_{[L][L']}^{(\text{in})}(E, r) + h_{\lambda}^{(+)}(kr) F_{[L][L']}^{(\text{out})}(E, r), \quad (11)$$

where the incoming and outgoing hyperspherical waves described by the Riccati-Hankel functions,

$$h_{\lambda}^{(\pm)}(kr) \xrightarrow{|kr| \rightarrow \infty} \mp i \exp[\pm i(kr - \lambda\pi/2)], \quad (12)$$

are included explicitly. The matrices  $F_{[L][L']}^{(\text{in/out})}(E, r)$  are new unknown functions. In the theory of ordinary differential equations, this way of finding solution is known as the variation parameters method (see, for example, Ref.

[17]).

Since instead of one unknown matrix  $u_{[L][L']}$  we introduce two unknown matrices  $F_{[L][L']}^{(\text{in/out})}$ , they cannot be independent. We therefore can impose an arbitrary condition that relates them to each other. As such condition, it is convenient to choose the following equation

$$h_{\lambda}^{(-)}(kr) \partial_r F_{[L][L']}^{(\text{in})}(E, r) + h_{\lambda}^{(+)}(kr) \partial_r F_{[L][L']}^{(\text{out})}(E, r) = 0, \quad (13)$$

which is standard in the variation parameters method and is called the Lagrange condition. Substituting the ansatz (11) into the hyperradial equation (9) and using

the condition (13), we obtain the following system of first order equations for these unknown matrices

$$\begin{cases} \partial_r F_{[L][L']}^{(\text{in})} = -\frac{h_{\lambda}^{(+)}}{2ik} \sum_{[L'']} V_{[L][L'']} \left[ h_{\lambda''}^{(-)} F_{[L''] [L']}^{(\text{in})} + h_{\lambda''}^{(+)} F_{[L''] [L']}^{(\text{out})} \right], \\ \partial_r F_{[L][L']}^{(\text{out})} = +\frac{h_{\lambda}^{(-)}}{2ik} \sum_{[L'']} V_{[L][L'']} \left[ h_{\lambda''}^{(-)} F_{[L''] [L']}^{(\text{in})} + h_{\lambda''}^{(+)} F_{[L''] [L']}^{(\text{out})} \right], \end{cases} \quad (14)$$

which are equivalent to the second order Eq. (9). The regularity of a physical wave function at  $r = 0$  implies the following boundary conditions

$$F_{[L][L']}^{(\text{in})}(E, 0) = F_{[L][L']}^{(\text{out})}(E, 0) = \delta_{[L][L']}. \quad (15)$$

With these conditions, the columns of the matrix  $u_{[L][L']}(E, r)$  are not only regular but linearly independent as well. Therefore any regular column  $\phi_{[L]}(E, r)$  obeying Eq. (9), can be written as a linear combination of the columns of matrix  $u_{[L][L']}(E, r)$ . In other words, the matrix  $u_{[L][L']}(E, r)$  is a complete basis for the regular solutions.

At large hyperradius where the potentials vanish, i.e.

$$V_{[L][L']}(r) \xrightarrow{r \rightarrow \infty} 0, \quad (16)$$

the right-hand sides of Eqs. (14) should tend to zero and therefore the matrices  $F_{[L][L']}^{(\text{in/out})}(E, r)$  converge to the energy-dependent constants,

$$f_{[L][L']}^{(\text{in/out})}(E) = \lim_{r \rightarrow \infty} F_{[L][L']}^{(\text{in/out})}(E, r), \quad (17)$$

that by analogy with the two-body case can be called the Jost matrices. The convergency of these limits, however, depends on the choice of the energy  $E$  and on how fast the potential matrix  $V_{[L][L']}(r)$  vanishes when  $r \rightarrow \infty$ .

When the energy is real and positive (scattering states), the vanishing of the right hand sides of Eqs. (14) at large distances is completely determined by the behaviour of  $V_{[L][L']}(r)$ . It can be shown that in such a case the limits (17) exist if  $V_{[L][L']}(r)$  vanishes faster than  $1/r$ .

With negative and complex energies there is a technical complication. The problem is that one of the Riccati-Hankel functions on the right hand side of Eqs. (14) is always exponentially diverging. Therefore, if at large distances the potential matrix vanishes not fast enough, the convergency of (17) is not achieved. This problem can be easily circumvented by using different path to the far-away point (see Fig. 2).

This is known as the the complex rotation of the coordinate. All the details concerning convergency of the limits (17) and the use of complex rotation for this purpose can be found in Refs. [14, 18, 19, 20, 21, 22].

As was said before, the columns of the matrix function  $u_{[L][L']}(E, r)$  constitute the regular basis using which we can construct a physical solution  $\phi_{[L]}(E, r)$  with given boundary conditions at infinity,

$$\phi_{[L]}(E, r) = \sum_{[L']} u_{[L][L']}(E, r) C_{[L']}, \quad (18)$$

where  $C_{[L]}$  are the combination coefficients.

The spectral points  $E_n$  (bound and resonant states) are those at which the physical solution has only outgoing waves in its asymptotics, i.e. when

$$\sum_{[L']} f_{[L][L']}^{(\text{in})}(E_n) C_{[L']} = 0. \quad (19)$$

This homogeneous system has a non-trivial solution if and only if

$$\det f_{[L][L']}^{(\text{in})}(E_n) = 0, \quad (20)$$

which determines the spectral energies  $E_n$ . As can be easily shown [23], the  $S$ -matrix is given by

$$S(E) = f_{\ell}^{(\text{out})}(E) \left[ f_{\ell}^{(\text{in})}(E) \right]^{-1} \quad (21)$$

and therefore at the energies  $E_n$  it has poles.

### III. TWO-BODY POTENTIALS

In our calculations, we used local two-body potentials describing the interaction between two neutrons,  $\Lambda$  and neutron, and between two  $\Lambda$ -particles. For all these potentials, we used the same functional form, namely,

$$U(\rho) = \left[ A_1(\rho) - \frac{1+P^\sigma}{2} A_2(\rho) - \frac{1-P^\sigma}{2} A_3(\rho) \right] \left[ \frac{\beta}{2} + \frac{1}{2}(2-\beta)P^r \right], \quad (22)$$

$$A_n(\rho) = W_n \exp(-a_n \rho^2), \quad n = 1, 2, 3, \quad (23)$$

where  $P^\sigma$  and  $P^r$  are the permutation operators in the spin and configuration spaces, respectively. The form of  $U(\rho)$  as well as the parameters were taken from Ref. [24]. In order to explore how sensitive the positions of the three-body resonances are to the choice of underlying two-body potentials, we did the calculations with three different sets of parameters for the  $\Lambda n$ -potential. All the sets of parameters we used, are given in Table I.

#### IV. THE MINIMAL APPROXIMATION

The system (14) consists of infinite number of equations. For any practical calculation, one has to truncate it somewhere. Before going any further, it is very logical to try the simplest approximation, namely, when only the first terms of the sums on the right hand sides of Eqs. (14) are retained. This corresponds to the minimal ( $n = 0$ ) value of the grand orbital number (6) and is called the hypercentral approximation,  $[L] = [L_{\min}]$ . We assume that the two-body subsystems are in the  $S$ -wave states ( $\ell_1 = \ell_2 = 0$ ), which means that

$$\lambda = \lambda_{\min} = \frac{3}{2}.$$

So, in the minimal approximation, instead of the infinite system (14), we remain with only one equation,

$$\left[ \partial_r^2 + k^2 - \frac{\lambda_{\min}(\lambda_{\min} + 1)}{r^2} \right] u(E, r) = 2M \langle U \rangle u(E, r), \quad (24)$$

where all unnecessary subscripts are dropped, and the brackets on the right hand side mean the following integration

$$\langle U \rangle(r) = \int \Phi_{[L_{\min}]}^{jjz*}(\omega) (U_{12} + U_{13} + U_{23}) \Phi_{[L_{\min}]}^{jjz}(\omega) d\omega. \quad (25)$$

From the mathematical point of view, Eq. (24) looks exactly like the two-body radial Schrödinger equation. The only difference is that the angular momentum is not an integer number.

The explicit expression for the integral (25) is given in the Appendix. The hypercentral potentials  $\langle U \rangle$  for the systems  $\Lambda nn$  and  $\Lambda \Lambda n$  are shown in Figs 3 and 4. With these hyperradial potentials the corresponding differential equations determining the three-body Jost functions, were numerically solved with complex values of the energy. The results of these calculations are discussed next.

#### V. NUMERICAL RESULTS AND CONCLUSION

When looking for zeros of the three-body Jost functions, we found that there were no such zeros at real negative energies. In other words, neither the system  $\Lambda nn$  nor  $\Lambda \Lambda n$  have bound states.

The only zeros we found were located on the unphysical sheet of the energy surface, in the resonance domain. The resonance energies are given in Tables II and III and shown in Fig. 5. As is seen, the positions of these resonances depend on the choice of the  $\Lambda n$  potential. For the choice “C”, the resonances become sub-threshold.

In order to estimate how far our three-body systems are from being bound, we artificially increased the depths of the potentials by multiplying them by a scaling factor. When this factor was increased from 1 upwards, the Jost function zeros moved towards the origin of the energy surface. At the value of approximately 1.5, the zeros crossed the threshold and moved onto the real negative axis. In other words, the bound states can appear if the potential strength is increased by  $\sim 50\%$ .

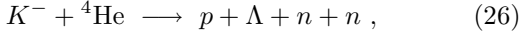
The fact that we did not find bound  $\Lambda nn$  or  $\Lambda \Lambda n$  states is not surprising at all. As is shown in Refs. [3, 5], the system  $\Lambda \Lambda N$  in the state with the three-body isospin 1 and spin  $s = 1/2$  is not bound even when the virtual processes of  $\Lambda - \Sigma$  conversion are taken into account, although this conversion increases the attraction in the system. Simple but convincing argumentation of Ref. [25] leads us to the conclusion that the  $\Lambda \Lambda n$  system also cannot be bound. Indeed, the system  $\Lambda \Lambda n$  is a “mirror” image of  $\Lambda nn$ , where the  $\Lambda$  and  $n$  replace each other. This means that the potential term  $U = U_{nn} + U_{\Lambda n} + U_{\Lambda n}$  of the three-body Hamiltonian is replaced with  $U = U_{\Lambda \Lambda} + U_{\Lambda n} + U_{\Lambda n}$ . Since the attraction of  $U_{\Lambda \Lambda}$  is weaker than that of  $U_{nn}$ , we may conclude that the system  $\Lambda \Lambda n$  has less chances to be bound than the system  $\Lambda nn$ . The calculations performed in Refs. [24, 26, 27, 28], show that even the heavier hypernucleus  ${}^4_{\Lambda \Lambda} \text{H}$  (i.e. the system  $\Lambda \Lambda pn$ ) is bound very weakly, if bound at all.

Multiplying the two-body potentials by an appropriate scaling factor, we can always generate an artificial three-body bound state, i.e. a pole of the  $S$ -matrix on the physical sheet of the  $E$ -surface at a negative energy. Apparently this pole cannot disappear when the scaling factor returns to its natural value of 1. The pole simply moves via the threshold onto the unphysical sheet. Since both the systems we consider, are not far from being bound, their corresponding poles cannot be far away from the threshold energy. And indeed we located them

at low energies.

What we found is, of course, an estimate. But it clearly shows that there are near-threshold resonances of the systems  $\Lambda nn$  and  $\Lambda\Lambda n$ . Actual location of the poles most probably is more close to the threshold energy. An inclusion of the channels  $\Lambda N - \Sigma N$  and  $\Lambda\Lambda - \Xi N$  would definitely increase the attraction in our systems (see Ref. [29]) and this would make the widths of the resonances smaller.

As we have demonstrated, the positions of the resonances strongly depend on the choice of the two-body potentials. If such resonances are observed experimentally, they may serve as an additional instrument for constructing adequate  $YN$  and  $YY$  potentials. There are many possible reactions where the three-body resonances  $\Lambda nn$  and  $\Lambda\Lambda n$  may manifest themselves. As an example, we can mention the inelastic collision of the  $K^-$  meson with the  $\alpha$  particle,



that produces a proton and the system we are looking for.

If a short-lived cluster  $\Lambda nn$  is formed in the final state of this collision, it should be seen in the corresponding two-body kinematics  $p - \Lambda nn$ . The processes of the type (26) fall under the experimental programme AMADEUS [30] (in the INFN, Italy) and, in principle, this reaction could be thoroughly studied.

## APPENDIX

Hypercentral potential (25) consists of the three terms

$$\langle U \rangle = \langle U_{12} \rangle + \langle U_{13} \rangle + \langle U_{23} \rangle, \quad (27)$$

where  $U_{ij}$  is the two-body potential acting between particles  $i$  and  $j$ . As was mentioned above, we can consider both  $nn\Lambda$  and  $\Lambda\Lambda n$  systems in a unified way. Let 1 and 2 be the identical particles, i.e. the  $nn$  or  $\Lambda\Lambda$  pair, and 3 be the remaining  $\Lambda$ -particle or neutron, respectively.

The six-dimensional volume element is

$$\begin{aligned} d\vec{r}_1 d\vec{r}_2 &= r_1^2 r_2^2 dr_1 dr_2 \sin \theta_{\vec{r}_1} d\theta_{\vec{r}_1} d\varphi_{\vec{r}_1} \sin \theta_{\vec{r}_2} d\theta_{\vec{r}_2} d\varphi_{\vec{r}_2} \\ &= r^5 dr \frac{1}{4} \sin^2(2\alpha) d\alpha \sin \theta_{\vec{r}_1} d\theta_{\vec{r}_1} d\varphi_{\vec{r}_1} \sin \theta_{\vec{r}_2} d\theta_{\vec{r}_2} d\varphi_{\vec{r}_2} \\ &= r^5 dr d\omega. \end{aligned}$$

Therefore in the five-dimensional integral (25) the volume element is

$$d\omega = \frac{1}{4} \sin^2(2\alpha) \sin \theta_{\vec{r}_1} \sin \theta_{\vec{r}_2} d\alpha d\theta_{\vec{r}_1} d\varphi_{\vec{r}_1} d\theta_{\vec{r}_2} d\varphi_{\vec{r}_2}. \quad (28)$$

Since we assume that  $\ell_1 = \ell_2 = 0$  and  $L = L_{\min} = 0$ , the sum (7) is reduced to a single term,

$$\Phi_{[L_{\min}]}^{jj_z}(\omega) = Y_{[L_{\min}]}(\omega) \chi_{[s]}, \quad (29)$$

where the quantum numbers  $jj_z$  coincide with  $s\sigma$ . The two-body spin  $s_{12}$  of the identical pair in the  $S$ -wave state must be zero. As a result the three-body spin  $s$  is always  $1/2$ . The hyperspherical harmonics  $Y_{[L_{\min}]}(\omega)$  is trivial (independent of the angles),

$$Y_{[L_{\min}]}(\omega) \equiv \pi^{-3/2}, \quad (30)$$

which means that the action of the permutation operators  $P^r$  for all three terms in Eq. (27) is also trivial: its eigenvalue is 1,

$$P_{ij}^r Y_{[L_{\min}]} = Y_{[L_{\min}]} , \quad ij = \{12\}, \{13\}, \{23\}. \quad (31)$$

The spin permutation operator  $P_{12}^\sigma$  for the identical pair  $\{12\}$  changes the sign of  $\chi_{[s]}$ ,

$$P_{12}^\sigma \chi_{[s]} = -\chi_{[s]}, \quad (32)$$

because  $s_{12} = 0$  in  $[s] = ((s_1 s_2) s_{12} s_3) s\sigma$ . For the other two pairs, its action is a bit more complicated. Indeed, recoupling the spins,

$$|((s_1 s_2) s_{12} s_3) s\sigma\rangle = \sum_{s_{31}} |((s_3 s_1) s_{31} s_2) s\sigma\rangle \langle((s_3 s_1) s_{31} s_2) s\sigma| ((s_1 s_2) s_{12} s_3) s\sigma\rangle$$

$$\begin{aligned}
&= |((s_3 s_1) 0 s_2) s \sigma\rangle \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\} + \sqrt{3} |((s_3 s_1) 1 s_2) s \sigma\rangle \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{array} \right\} \\
&= -\frac{1}{2} |((s_3 s_1) 0 s_2) s \sigma\rangle + \frac{\sqrt{3}}{2} |((s_3 s_1) 1 s_2) s \sigma\rangle ,
\end{aligned}$$


---

we find that

$$\chi_{[s]}^+ P_{13}^\sigma \chi_{[s]} = \frac{1}{2} . \quad (33)$$

Similarly, it is easy to find for the pair {23} that

$$\chi_{[s]}^+ P_{23}^\sigma \chi_{[s]} = \frac{1}{2} . \quad (34)$$


---

When inserting the potentials  $U_{ij}$  given by Eq. (22), into the integral (25), we should use the following interparticle distances (see Fig. 1),

$$\rho_{12} = r \sqrt{\frac{M}{\mu_1}} \cos \alpha , \quad (35)$$

$$\rho_{13} = r \sqrt{\frac{M}{\mu_2} \sin^2 \alpha + \frac{M}{4\mu_1} \cos^2 \alpha - \frac{M}{2\sqrt{\mu_1 \mu_2}} \sin(2\alpha) \cos \theta_{\vec{r}_2}} , \quad (36)$$

$$\rho_{23} = r \sqrt{\frac{M}{\mu_2} \sin^2 \alpha + \frac{M}{4\mu_1} \cos^2 \alpha + \frac{M}{2\sqrt{\mu_1 \mu_2}} \sin(2\alpha) \cos \theta_{\vec{r}_2}} . \quad (37)$$


---

Since the particles 1 and 2 are identical the interactions  $U_{13}$  and  $U_{23}$  are the same. Moreover, according to Eqs. (33, 34), the product  $\chi_{[s]}^+ U_{13} \chi_{[s]}$  has the same dependence on  $\rho_{13}$  as  $\chi_{[s]}^+ U_{23} \chi_{[s]}$  depends on  $\rho_{23}$ . Actually, they depend on  $\rho_{13}^2$  and  $\rho_{23}^2$  because all the terms in the potential (22) are of the Gaussian form. Comparing Eqs. (36) and

(37), we see that the integrands for  $\langle U_{13} \rangle$  and  $\langle U_{23} \rangle$  differ only in the sign of the power of the exponential factors corresponding to the last terms of  $\rho_{13}^2$  and  $\rho_{23}^2$ . This difference however has no effect on the integrals. Indeed, the integration over  $\theta_{\vec{r}_2}$ ,

$$\begin{aligned}
\int_0^\pi \exp(\pm f \cos \theta_{\vec{r}_2}) \sin \theta_{\vec{r}_2} d\theta_{\vec{r}_2} &= \int_{-1}^1 \exp(\pm f t) dt \\
&= \frac{1}{f} (e^f - e^{-f}) = \frac{2}{f} \sinh(f) ,
\end{aligned}$$

gives the same result for both signs. Therefore  $\langle U_{13} \rangle = \langle U_{23} \rangle$  and hence

$$\langle U \rangle = \langle U_{12} \rangle + 2\langle U_{13} \rangle . \quad (38)$$

Performing trivial integrations over  $\varphi_{\vec{r}_1}$ ,  $\varphi_{\vec{r}_2}$ ,  $\theta_{\vec{r}_1}$ , and  $\theta_{\vec{r}_2}$  (trivial in the case of  $\langle U_{12} \rangle$ ), we obtain the following expressions for the terms of the hypercentral potential (38),

$$\langle U_{12} \rangle = \frac{4}{\pi} \int_0^{\pi/2} d\alpha \sin^2(2\alpha) \left[ W_1^{\{12\}} \exp(-a_1^{\{12\}} \eta r^2) - W_3^{\{12\}} \exp(-a_3^{\{12\}} \eta r^2) \right] , \quad (39)$$

$$\begin{aligned}
\langle U_{13} \rangle = & \frac{2}{\pi} \int_0^{\pi/2} d\alpha \sin^2(2\alpha) \left[ W_1^{\{13\}} \exp\left(-a_1^{\{13\}} \zeta r^2\right) s\left(a_1^{\{13\}} \xi r^2\right) \right. \\
& - \frac{3}{4} W_2^{\{13\}} \exp\left(-a_2^{\{13\}} \zeta r^2\right) s\left(a_2^{\{13\}} \xi r^2\right) \\
& \left. - \frac{1}{4} W_3^{\{13\}} \exp\left(-a_3^{\{13\}} \zeta r^2\right) s\left(a_3^{\{13\}} \xi r^2\right) \right] ,
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
\eta(\alpha) &= \frac{M}{\mu_1} \cos^2 \alpha , \\
\zeta(\alpha) &= \frac{M}{\mu_2} \sin^2 \alpha + \frac{M}{4\mu_1} \cos^2 \alpha , \\
\xi(\alpha) &= \frac{M}{2\sqrt{\mu_1\mu_2}} \sin(2\alpha) , \\
s(f) &= \frac{1}{f} (e^f - e^{-f}) .
\end{aligned}$$

The parameters  $W_n^{\{ij\}}$  and  $a_n^{\{ij\}}$ , where the symbol  $\{ij\}$  means a choice of the pair of interacting particles, are given in Table I. For each (complex) value of the hyperradius  $r$ , which was needed in our calculations, the integrals (39) and (40) were evaluated numerically.

### Acknowledgments

Financial support from Deutsche Forschungsgemeinschaft (DFG grant no 436 RUS 113/761/0-2) is greatly appreciated.

- 
- [1] Gal A., The hypernuclear physics heritage of Dick Dalitz (1925-2006). arXiv: physics/0701019 (2007).
  - [2] Nogga A., Kamada H., Glöckle W., The hypernuclei  ${}^4_\Lambda\text{He}$  and  ${}^4_\Lambda\text{H}$ : challenges for modern hyperon-nucleon forces. Phys. Rev. Lett., **88**, 172501 (2002).
  - [3] Miyagawa K., Kamada H., Glöckle W., Stoks V., Properties of the bound  $\Lambda(\Sigma)NN$  system and hyperon-nucleon interactions. Phys. Rev., **C51**, 2905-2913 (1995).
  - [4] Nemura H., Akaishi Y., Suzuki Y., Ab initio approach to  $s$ -shell hypernuclei  ${}^3_\Lambda\text{H}$ ,  ${}^4_\Lambda\text{H}$ ,  ${}^4_\Lambda\text{He}$ , and  ${}^5_\Lambda\text{He}$  with a  $\Lambda N - \Sigma N$  interaction. Phys. Rev. Lett., **89**, 142504 (2002).
  - [5] Garcilazo H., Fernández-Caramés T., Valcarce A.,  $\Lambda NN$  and  $\Sigma NN$  systems at threshold. Phys. Rev. **C75**, 034002, 1-10 (2007).
  - [6] Garcilazo H., Valcarce A., Fernández-Caramés T.,  $\Lambda NN$  and  $\Sigma NN$  systems at threshold: II. The effect of D waves, arXiv:0708.0199v1, hep-ph, (2007).
  - [7] Belyaev V. B., Lectures on the theory of few-body systems. Springer-Verlag, Berlin Heidelberg (1990)
  - [8] Alexander G., Karshon U., Shapira A., Yekutieli G., Engelmann R., Filthuth H., Lughofer W., Study of the  $\Lambda - N$  system in low-energy  $\Lambda - p$  elastic scattering. Phys. Rev., **173**, 1452-1460 (1968).
  - [9] Ansari H. H., Shoeb M., Rahman Khan M. Z., Low-energy  $\Lambda p$  scattering and  $p$ -shell hypernuclei. J. Phys. G: Nucl. Phys., **12**, 1369-1377 (1986).
  - [10] Afnan I. R., Gibson B. F., Resonances in  $\Lambda d$  scattering and the  $\Sigma$  hypertriton. Phys. Rev. C, **47**, 1000-1012 (1993).
  - [11] Kahana D. E., Kahana S. H., Millener D. J., Resonant state in  ${}^4_\Lambda\text{He}$ . Phys. Rev. C **68**, 037302 (2003).
  - [12] Schaffner-Bielich J., Hypernuclear physics and compact stars. arXiv: astro-ph/0703113v1 (2007).
  - [13] Alt E. O., Grassberger P., Sandhas W., Reduction of the three-particle collision problem to multi-channel two-particle Lippmann-Schwinger equations. Nucl. Phys., **B2**, 167-180 (1967).
  - [14] Rakityansky S. A., Sofianos S. A. and Amos K., A method for calculating the Jost function for analytic potentials. Nuovo Cim. B **111**, 363-378 (1996).
  - [15] Schmid E. W. and Ziegelmann H., The Quantum mechanical Three-Body Problem, p. 183, Pergamon Press, Oxford, 1974
  - [16] Fabre de la Ripelle M., The potential harmonic expansion method. Ann. Phys., **147**, 281-320 (1983).
  - [17] L. Brand, *Differential and Difference Equations*, John Wiley & Sons, Inc., New York, (1966).
  - [18] Sofianos S. A. and Rakityansky S. A., Exact method for locating potential resonances and Regge trajectories. J. Phys. A: Math. Gen. **30** 3725-3737 (1997).
  - [19] Sofianos S. A., Rakityansky S. A., Vermaak G. P., Sub-threshold resonances in few-neutron systems. J. Phys. G: Nucl. Part. Phys., **23**, 1619-1629 (1997).
  - [20] Rakityansky S. A. and Sofianos S. A., Jost function for coupled partial waves. J. Phys. A: Math. Gen. **31** 5149-5175 (1998).
  - [21] Sofianos S. A., Rakityansky S. A. and Massen S. E., Jost function for singular potentials. Phys. Rev. A **60**, 337-343 (1999).
  - [22] Rakityansky S. A. and Sofianos S. A., Jost function for coupled channels. Few-Body Systems Suppl. **10**, 93-96 (1999).
  - [23] Rakityansky S. A. and Elander N., Analyzing the contribution of individual resonance poles of the  $S$ -matrix to two-channel scattering. Int. J. Quant. Chem., **106**, 1105-1129 (2006).
  - [24] Nemura H., Suzuki Y., Fujiwara Y., Nakamoto C., Study of light  $\Lambda$ - and  $\Lambda\Lambda$ -hypernuclei with the stochastic varia-

	$nn$	$\Lambda\Lambda$	$\Lambda n$ (A)	$\Lambda n$ (B)	$\Lambda n$ (C)
$W_1$ (MeV)	200.0	200.0	200.0	600.0	5000
$W_2$ (MeV)	178.0	0	106.5	52.61	47.87
$W_3$ (MeV)	91.85	130.8	118.65	66.22	61.66
$a_1$ (fm $^{-2}$ )	1.487	2.776	1.638	5.824	18.04
$a_2$ (fm $^{-2}$ )	0.639	0	0.7864	0.6582	0.6399
$a_3$ (fm $^{-2}$ )	0.465	1.062	0.7513	0.6460	0.6325
$\beta$	1	1	1	1	1

TABLE I: Parameters of the potential (22) for the pairs  $nn$ ,  $\Lambda\Lambda$ , and  $\Lambda n$ . For the system  $\Lambda n$ , three different sets of parameters (denoted as A, B, and C) are given. All the parameters are taken from Ref. [24].

$\Lambda n$ -potential	A	B	C
$E_0$ , (MeV)	$0.551 - \frac{i}{2}4.698$	$0.456 - \frac{i}{2}4.885$	$-0.149 - \frac{i}{2}5.783$

TABLE II: Complex resonance energies  $E_0 = E_r - \frac{i}{2}\Gamma$  for the system  $\Lambda nn$  with the three choices of  $\Lambda n$ -potential.

- tional method and effective  $\Lambda N$  potentials. Prog. Theor. Phys., **103**, 929-958 (2000).
- [25] Tang Y. C., Herndon R. C., Existence of light double hypernuclei. Phys. Rev. Lett., **14**, 991-995 (1965).
- [26] Filikhin I. N., Gal A., Light  $\Lambda\Lambda$  hypernuclei and the onset of stability for  $\Lambda\Xi$  hypernuclei. Phys. Rev. **C65**, 041001(R), 1-4 (2002).
- [27] Nemura H., Akaishi Y., Myint K. S., Stochastic variational search for  ${}_{\Lambda\Lambda}^4\text{H}$ . Phys. Rev. **C 67**, 051001(R), 1-4 (2003).
- [28] Gal A.,  $\Lambda\Lambda$  hypernuclei and stranger systems. Nucl. Phys., **A754**, 91c-102c (2005).
- [29] Hiyama E., Weakly bound states in light hypernuclei. Few-Body Systems, **34**, 79-84 (2004).
- [30] Oton Vazquez Doce, The AMADEUS experiment: study of the kaonic nuclear clusters at DAΦNE, The 20th European Conference on Few-Body Problems in Physics, EFB 20, Pisa, Italy, September 10-14, 2007, Book of Abstracts, p. 94.

$\Lambda n$ -potential	A	B	C
$E_0$ , (MeV)	$0.096 - \frac{i}{2}8.392$	$0.034 - \frac{i}{2}8.438$	$-0.552 - \frac{i}{2}8.681$

TABLE III: Complex resonance energies  $E_0 = E_r - \frac{i}{2}\Gamma$  for the system  $\Lambda\Lambda n$  with the three choices of  $\Lambda n$ -potential.

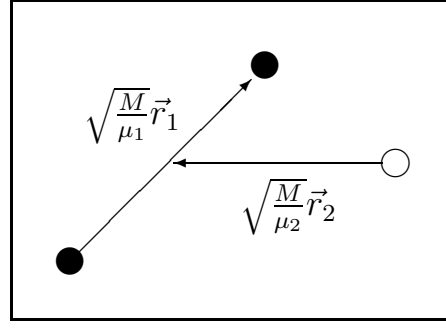


FIG. 1: Jacobi vectors defining the spatial configuration of a three-body system of two identical (filled circles) and one different (open circle) particles.

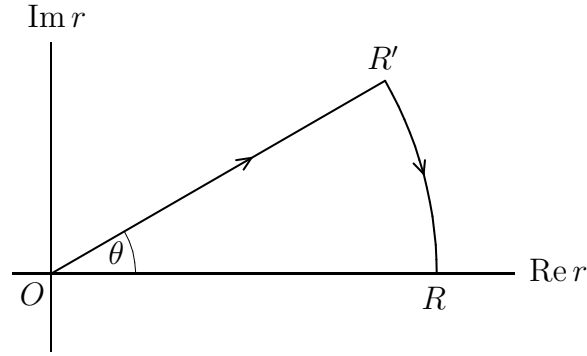


FIG. 2: Deformed contour for integrating Eqs. (14) from  $r = 0$  to  $r = R$  when the energy is complex.

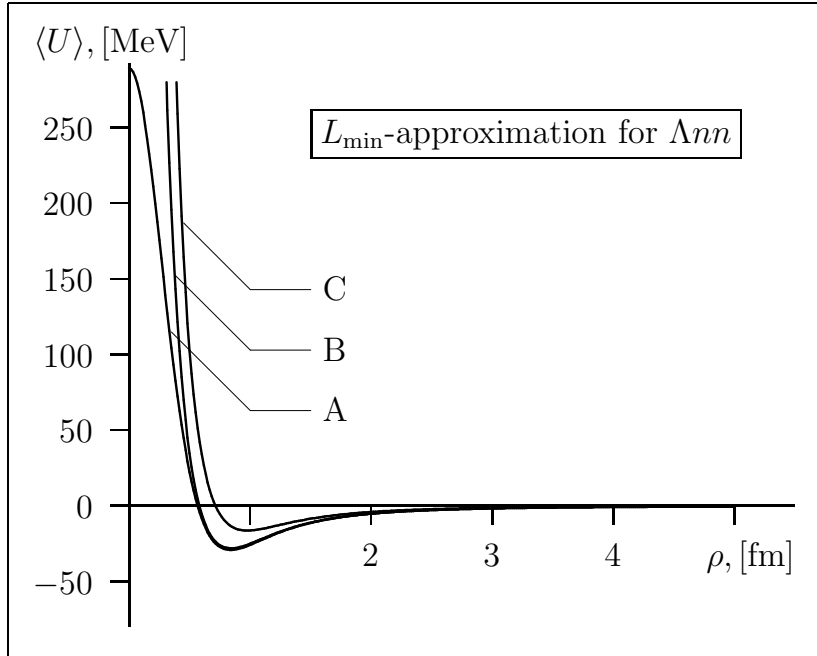


FIG. 3: The hypercentral potential given by Eq. (25) for the system  $\Lambda nn$  with the three choices (A, B, and C) of the  $\Lambda n$  interaction.

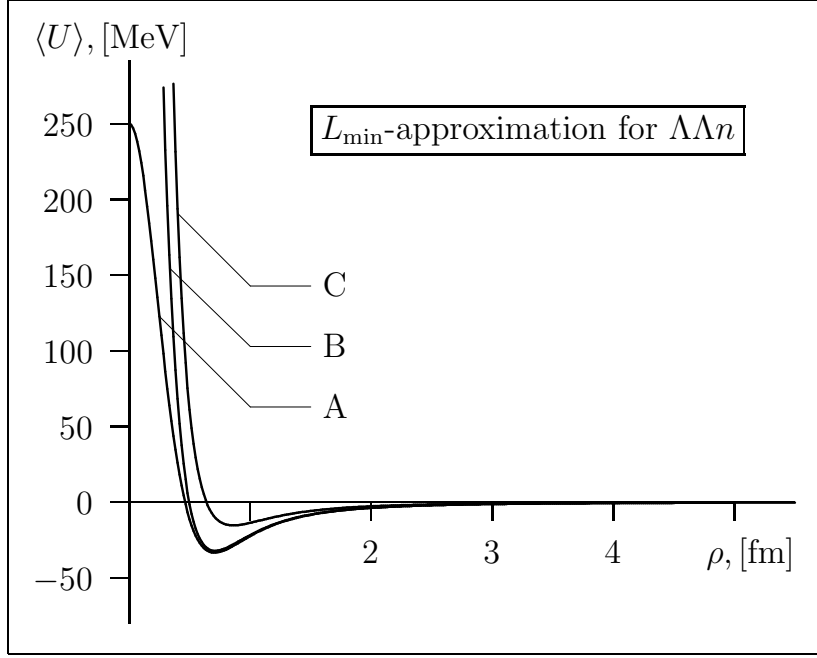


FIG. 4: The hypercentral potential given by Eq. (25) for the system  $\Lambda\Lambda n$  with the three choices (A, B, and C) of the  $\Lambda n$  interaction.

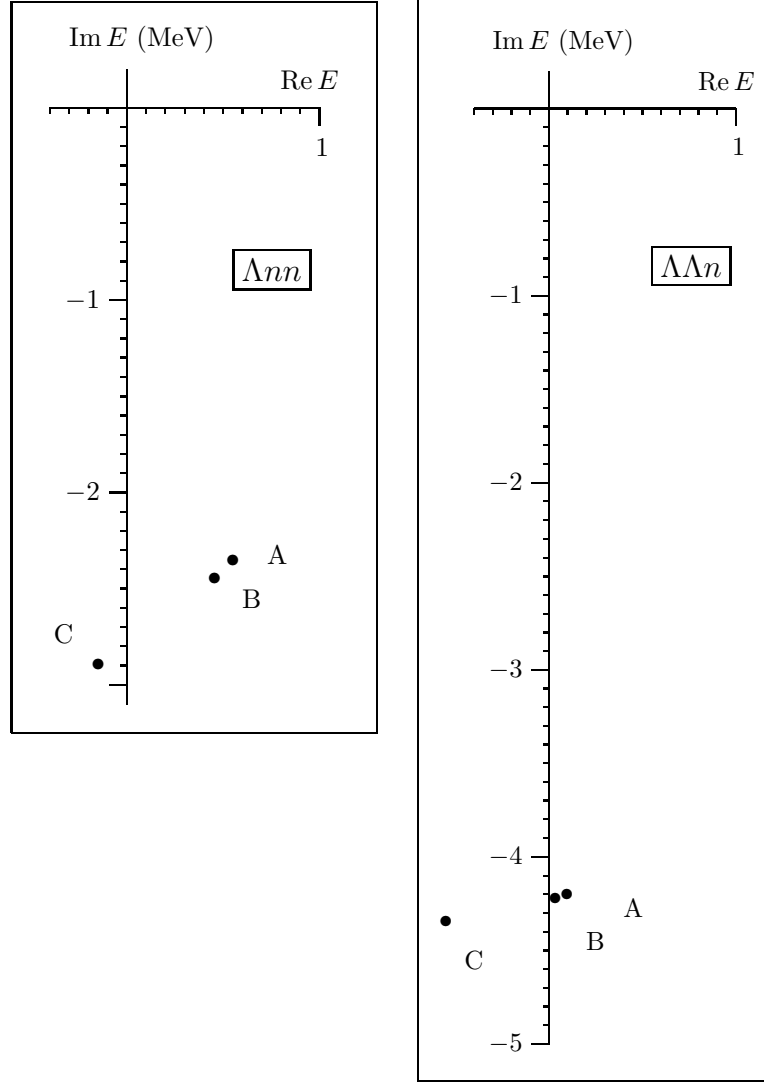


FIG. 5: Resonance points for the systems  $\Lambda nn$  and  $\Lambda\Lambda n$  found on the unphysical sheet of the energy surface with the three sets (A, B, and C) of parameters of the  $\Lambda n$ -potential given in Table I.